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On the free-boundary for an elliptic inverse nonlocal problem arising in the nuclear fusion. A numerical approach

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Abstract

We consider a mathematical model related to the stationary regime of a plasma magnetically confined in a Stellarator device in the nuclear fusion. The mathematical problem may be reduced to an nonlinear elliptic inverse nonlocal two dimensional free–boundary problem. The nonlinear terms involving the unknown functions of the problem and its rearrangement. Our main goal is to determinate the existence and the estimate on the location and size of region where the solution is nonnegative almost everywhere (corresponding to the plasma region in the physical model).

1 Introduction

One of the main difficulties of plasma magnetic confinement for controlled nuclear fusion is to determine conditions on the magnetic field and the current density which prevents plasma from contacting the walls of the fusion camera. It is a very important to know how is the plasma (its boundary) away from the walls of the reactor. To determinate an estimate of this distance and to show some numerical simulation for a given mathematical model will be the aim of this work (see, e.g. [3] for the mathematical model, [5, 7] for the estimate of the plasma and [10, 2] for some numerical analysis). We consider the two dimensional inverse nonlocal free–boundary problem (with only one unknown) obtained in [3] after to derivating the integral conditions over the level set that determines the magnetic surface. So, the problem is to find one weak solution $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ for

$$(\mathcal{P}_*) \left\{ \begin{array}{c} -\Delta u\left(x\right) = a\left(x\right) G_u\left(u\left(x\right)\right) + H\left(u\left(x\right), b_{*u}\right) \text{ in } \Omega, \\ u\left(x\right) - \gamma \in H_0^1(\Omega) \quad \text{on } x \in \partial\Omega \end{array} \right. \tag{1}$$

with Ω is an open regular set and where the real functions G_u and H (defined in the next section) are, in particular, depending of the space, the solution u and of its rearrangement (we introduce the notion and some properties of decreasing and relative rearrangement of a function in the next section. For more details see e.g. [3] and its references, [6] and [9]). The problem $(\mathcal{P}_*)$ is an inverse type, due to fact that the nonlinear term associated to $G_u(\cdot)$ is an unknown which will be determined by the unknown function u.

The main goal of this paper is to obtain the existence and the estimate on the location and size of plasma region $\Omega_p = \{x \in \Omega : u\left(x\right) \geq 0\}$ and the vacuum region $\Omega_v = \{x \in \Omega : u\left(x\right) < 0\}$ for the ideal Stellarator device. That is stated as follow:

Theorem 1 *Let Ω be a bounded open regular subset of $\mathbb{R}^2$ (with C^1 boundary $\partial\Omega$) and such that*

$$\exists\ x_0 \in \Omega \text{ verifying } R_p := \left(\frac{-4\gamma}{F_v \operatorname*{ess\,inf}_{x\in\Omega} a\left(x\right)} \right)^{\frac{1}{2}} < d\left(x, \partial\Omega\right)$$

Assume that $\operatorname*{ess\,inf}_{\Omega} a > 0$. *Let u be a weak solution of $(\mathcal{P}_*)$ such that u has not flat region and $F \in W^{1,\infty}(\operatorname{ess\,inf}_{x\in\Omega} u, \operatorname{ess\,sup}_{x\in\Omega} u)$. Then, if λ are small enough, we have that $\Omega_{R_p} := \{x \in \Omega : d\left(x, \partial\Omega\right) \geq R_p\} \subset \Omega_p = \{x \in \Omega : u\left(x\right) \geq 0\}$.*

In particular $\operatorname{meas}\{x \in \Omega\text{: } d\left(x, \partial\Omega\right) \geq R_p\} \leq |\Omega_p|$ *(d denotes the Euclidean distance).*

Analogously, we find a similar estimate for the location and size of the vacuum region $\Omega_v := \{x \in \Omega : u\left(x\right) < 0\}$:

Theorem 2 *Let Ω be an open bounded regular (with C^1 boundary $\partial\Omega$) subset of $\mathbb{R}^2$ and such that $\exists\ x_0 \in \Omega$ verifying $R_p < d\left(x, \partial\Omega\right)$, then there exists a positive number $\hat{R}$ such that $0 < \rho < \hat{R} \leq R_p + \rho$ for some $\rho > 0$ and that for any $\bar{x} \in \partial\Omega$ the segment $\bar{x} + r\mathbf{n}, 0 < r \leq \hat{R}$ belongs to Ω where $\mathbf{n}$ is the inward normal unit vector to $\partial\Omega$. Let u be a weak solution of $(\mathcal{P}_*)$ such that u has not flat region and $F \in W^{1,\infty}(\operatorname{ess\,inf}_{x\in\Omega} u, \operatorname{ess\,sup}_{x\in\Omega} u)$. Then, if λ are small enough we have that $\left\{x \in \Omega : d\left(x, \partial\Omega\right) \leq \hat{R} - \rho\right\} \subset \Omega_v = \{x \in \Omega : u\left(x\right) < 0\}$. In particular* $\operatorname{meas}\left\{x \in \Omega\text{: } d\left(x, \partial\Omega\right) \leq \hat{R} - \rho\right\} \leq |\Omega_p|$.

The structure of the rest of the paper is as follows. In Section 2 we introduce the notion of decreasing and relative rearrangement. We give some results proved in previous works concerning to the existence of solutions for problem $(\mathcal{P}_*)$ and some a priori estimate of solution. In Section 3 we prove the main results concerning to the existence and estimate the location and size of plasma region and vacuum region and we show some numerical results.

2 Previous results on the existence and a priori estimates for the solution of problem $(\mathcal{P}_*)$

We start recalling the notion of decreasing and relative rearrangement. Let Ω be a bounded and connected open measurable set of $\mathbb{R}^2$ (we assume a 2d–setting motivated by the physical modeling but the definitions and results that follows hold for any dimension $N > 1$). Given a measurable function $u : \Omega \to \mathbb{R}$, the *distribution function of* u is given by $m_u(\sigma) :=$meas$\{x \in \Omega : u(x) > \sigma\}$ (the Lebesgue measure of the set $\{x \in \Omega : u(x) > \sigma\}$ will be denoted by $|u > \sigma|$). It is well–know that the function $m_u(\cdot)$ is decreasing and right semicontinuous. We shall say that u has a flat region at the level σ if meas$\{x \in \Omega : u(x) = \sigma\}$ (denoting by $|u = \sigma|$) is strictly positive. The generalized inverse of m_u is called the *decreasing rearrangement* of u with respect to x and it is defined as the function $u_* : [0, |\Omega|] \to \bar{\mathbb{R}}$ such that $u_*(s) := \inf\{\sigma \in \mathbb{R} : m_u(\sigma) \leq s\}$ for all $s \in \Omega_*$, where $\Omega_* :=]0, |\Omega|[$ (see e. g. [3], [6] and [9] for more details about its definition and properties). We recall some properties: u_* is decreasing, $u_*(0) = ||u_+||_{L^\infty(\Omega)} = ess\sup_{x\in\Omega} u(x)$, u_* and u are equimeasurable, and the mapping $u \in L^p(\Omega)$ to $u_* \in L^p(\Omega_*)$ is a contraction for $1 \leq p \leq +\infty$. Moreover, if u has not flat regions, then m_u and u_* are continuous and $u_*(m_u(\sigma)) = \sigma$ (that is, $u_*^{-1} = m_u$). On the other hand, if $u \in W^{1,p}(\Omega)$, $1 \leq p \leq +\infty$, then $u_* \in W^{1,p}_{loc}(\Omega_*)$.

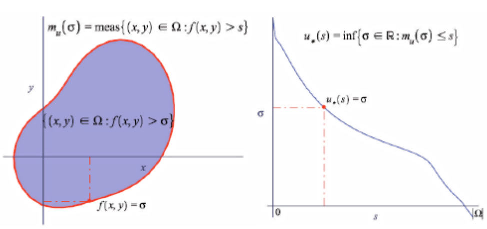

Figure 1: Numerical approximations of the decreasing rearrangement [8]

Given a measurable function $u : \Omega \to \mathbb{R}$, and $b \in L^p(\Omega)$ with $1 \leq p \leq \infty$, we define the function $w : \Omega_* \to \mathbb{R}$ as

$$w(s) = \int_{\{x\in\Omega : u(x) > u_*(s)\}} b(x)\, dx + \int_0^{s-|u(\cdot)>u_*(s)|} \left(b|_{\{x\in\Omega : u(x)=u_*(s)\}}\right)_*(\sigma)\, d\sigma.$$

The *relative rearrangement of* b *with respect to* u is the function $b_{*u} \in L^p(\Omega_*)$ defined by

$b_{*u}(s) := \frac{dw(s)}{ds} = \lim_{\sigma \to 0} \frac{(u+\sigma b)_*(s) - u_*(s)}{\sigma}$ for all $s \in \Omega_*$. Notice that by this definition, if u has not flat regions (implying $s - |u > u_*(s)| = 0$) then $b_{*u}(s) := \frac{d}{ds} \int_{\{x \in \Omega : u(x) > u_*(s)\}} b(x)\, dx$ for all $s \in \Omega_*$. Also, for any measurable function u, the mapping $b \in L^p(\Omega)$ to $b_{*u} \in L^p(\Omega_*)$ is a contraction for $1 \leq p \leq +\infty$ and in particular $||b_{*u}||_{L^\infty(\Omega_*)} \leq \|b\|_{L^\infty(\Omega)}$ (further details on the decreasing and relative rearrangement can be found, for instance, in [3], [6] and [9] and their references). See e. g. [10] and [2] for numerical approximations of the decreasing and relative rearrangement.

Before starting the results concerning to existence and a priori estimates of the solution of problem $(\mathcal{P}_*)$, we introduce the following useful convex cone $V(\Omega) := \{v \in H^1(\Omega) : \Delta v \in L^\infty(\Omega), v_{|\partial\Omega} \leq 0\}$. Given $u \in V(\Omega)$, the function G_u is defined as

$$G_u(s) := \left[F_v^2 - 2 \int_0^{s_+} p'(r)\, b_{*u}(|u > \sigma|)\, d\sigma \right]_+^{\frac{1}{2}} \tag{2}$$

In order to simplify the notation, we set

$$G_u(u(x)) = \left[F_v^2 - F_1(x, u(x), b_{*u}) \right]_+^{\frac{1}{2}} \tag{3}$$

where

$$F_1(x, u(x), b_{*u}) := 2 \int_{|u>0|}^{|u>u_+(x)|} [p(u_*)]'(\sigma)\, b_{*u}(\sigma)\, d\sigma. \tag{4}$$

Notice that the real functión G_u is a map from $V(\Omega)$ to real functión on $\mathbb{R}$, so it is also depending of u. We recalling the existence result and some a priori estimates given in [3] about the solution of problem $(\mathcal{P}_*)$.

Theorem 3 *Suppose that $\gamma \leq 0$ and $\inf_\Omega |a| > 0$. Then there exist $\Lambda_1, \Lambda_2 > 0$ such that if $\lambda \|b\|_{L^\infty(\Omega)} < \Lambda_1$ and $\Lambda_2 < \inf_\Omega |a| F_v$. Then, there exists $u \in V(\Omega)$ weak solution of the non local problem*

$$(\mathcal{P}_*) \begin{cases} -\Delta u(x) = a(x)\, G_u(u(x)) + H(u(x), b_{*u}) & \text{in } \Omega, \\ \qquad u(x) - \gamma \in H_0^1(\Omega) & \text{on } x \in \partial\Omega \end{cases} \tag{5}$$

satisfying also that $\operatorname{meas}\{x \in \Omega : \nabla u(x) = 0\} = 0$*; where the function H is given by*

$$H(u(x), b_{*u}) := p'(u(x))\, [b(x) - b_{*u}(|u > u(x)|)]. \tag{6}$$

Remark 4 *We can verify that if $s \leq 0$ then $G_u(s) = F_v > 0$ (it comes from (2)). If $u(x) \leq 0$ then $G_u(u(x)) = F_v$ (from (3) and (4)) and $H(u(x), b_{*u}) = 0$ from the definition of H and the hypotheses on p.*

Notice that, in the existence Theorem 3, assuming Λ_1 small enough (from λ small enough) we can define the positive number ν such that

$$\nu := \frac{\lambda |\Omega| \operatorname{osc}_\Omega b}{4\pi} < 1 \tag{7}$$

(with $\mathrm{osc}_\Omega b = \operatorname{ess\,sup}_{x\in\Omega} b - \operatorname{ess\,inf}_{x\in\Omega} b$). The existence of solution was proved in [3] by using a Galerkin type methods. In particular, recalling some result given in [3] and [7] and considering the fact that the current–carrying in an ideal Stellarator is $j \equiv 0$, we have the that the solution found in this way, it is such that verifies ([7] and its references) the following

Proposition 5 *For $\Lambda_1, \Lambda_2 > 0$ small enough, there exists a solution u of problem $(\mathcal{P}_*)$ given by Theorem 3 such that*

$$\|u_+\|_{L^\infty(\Omega)} \leq \frac{|\Omega|}{4\pi}\frac{\|a\|_{L^\infty(\Omega)}F_v}{(1-\nu)} := S,\ \|\Delta u\|_{L^\infty(\Omega)} \leq \frac{\|a\|_{L^\infty(\Omega)}F_v}{1-\nu} = \frac{4\pi}{|\Omega|}S,$$
$$\left\|\frac{du_{+*}}{ds}\right\|_{L^\infty(\Omega_*)} \leq \frac{1}{4\pi}\frac{\|a\|_{L^\infty(\Omega)}F_v}{1-\nu} = \frac{1}{|\Omega|}S$$

where ν is a positive number given by (7).

Corollary 6 *Given a solution u of problem $(\mathcal{P}_*)$ as in Proposition 5 we have that $0 \leq F_1(x, u(x), b_{*u}) \leq 2\lambda \|b\|_{L^\infty(\Omega)} S$, $|H(u, b_{*u})| \leq \lambda \mathrm{osc}_\Omega b S$ and thus $0 \leq G_u(u(x)) \leq F_v$, $G_u^2(u(x)) \geq \left[F_v^2 - 2\lambda \|b\|_{L^\infty(\Omega)} S\right]_+$.*

Corollary 7 *For this solution u, assuming λ small enough in order to have*

$$F_v^2 - 2\lambda \|b\|_{L^\infty(\Omega)} S > 0, \tag{8}$$

we have that $G_u(s) > 0$ for all $s \in [\inf_\Omega u, \sup_\Omega u]$ and $G_u(s)$ is Lipschitz*, i.e. there exists a positive number $C(\lambda)$ only dependents of λ, such that $|G_u(s) - G_u(\sigma)| \leq C(\lambda)|\sigma - s|$ for all $s, \sigma \in [\inf_\Omega u, \sup_\Omega u]$. Moreover $C(\lambda)$ goes to zero when λ goes to zero. On the other had there exist $\frac{d}{ds}G_u(s)$ and for all $x \in \Omega$,*

$$\left|\frac{1}{2}\left(G_u^2(u(x))\right)'\right| \leq \lambda \|b\|_{L^\infty(\Omega)} S, \quad |a(x)G_u(u(x)) + \tfrac{1}{2}\left(G_u^2(u(x))\right)' + b(x)p'(u(x))| \leq K$$

with $K := ||a||_{L^\infty(\Omega)}F_v + 2\lambda||b||_{L^\infty(\Omega)}S$. Finally,

$$H(u(x), b_{*u}) = \tfrac{1}{2}\left(G_u^2(u(x))\right)' + b(x)p'(u(x)).$$

3 Estimate on the location and size of the plasma region and the vacuum region.

We consider the following approach (see [5, 7]): *(i)* to give a condition for the existence of the free–boundary (i.e. $\Omega_p \neq \emptyset$), *(ii)* to verify that the solution u is supersolution for an auxiliary problem in a test balls in Ω, *(iii)* to give a suitable local subsolution $\underline{u}$ for this auxiliary problem satisfying the hypotheses of the comparison principle in the sense of Hopf [4] and finally *(iv)* to compare u with $\underline{u}$.

Lemma 8 *Let $B \subset \mathbb{R}^2$ and open ball of radius R centred at the origin and assume $\hat{a} \in L^\infty(B)$ be radially symmetric (i.e. $\hat{a}(x) = \tilde{a}(|x|)$ a.e. $x \in B$). Then, unique solution $u \in W^{2,p}(B)$, $p \in [1,\infty)$, to problem*

$$(PB)\begin{cases} -\Delta \underline{u} & = \hat{a} \quad \text{in } B \\ \underline{u} & = \gamma \quad \text{on } \partial B \end{cases}$$

satisfies: if $\int_0^R \left(\frac{1}{r}\int_0^r s\tilde{a}(s)\,ds\right)dr + \gamma = 0$, then $\underline{u}(0) = 0$, if $\int_0^R \left(\frac{1}{r}\int_0^r s\tilde{a}(s)\,ds\right)dr + \gamma < 0$, then $\underline{u}(0) < 0$ and if $\int_0^R \left(\frac{1}{r}\int_0^r s\tilde{a}(s)\,ds\right)dr + \gamma > 0$, then $\underline{u}(0) > 0$. Moreover, if $\int_0^r s\tilde{a}(s)\,ds \geq 0 \quad \forall r \in (0,R]$ then u decreases along the radius $r = |x|$.

Proof. The existence, regularity and uniqueness of $\underline{u}$, solution of (PB) is a well–known result (see for instance [1]). Moreover, $\underline{u}$ is a radial symmetric function in B (i.e. , so, $\underline{u}(x) = \tilde{\underline{u}}(|x|)$ $x \in B$) and is the unique solution to ordinary differential equation

$$\begin{cases} -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\tilde{\underline{u}}\right) & = \tilde{a}(r) \quad \text{in } 0 < r < R, \\ \tilde{\underline{u}}(R) & = \gamma \qquad \tilde{\underline{u}}(0) = 0. \end{cases}$$

By integration, we obtain the exact solution for previous ordinary differential equation $\tilde{\underline{u}}(r) = \int_0^R \left(\frac{1}{r}\int_0^r s\tilde{a}(s)\,ds\right)dr + \gamma$, $r \in [0,R]$ and from this and the fact that $\underline{u}(x) = \tilde{\underline{u}}(|x|)$ $x \in B$, we prove the lemma. □

Proof of Theorem 1. Let $x_0 \in \Omega$ such that $d(x_0, \partial\Omega) \geq R_p$ with $R_p = \left(\frac{-4\gamma}{F_v \operatorname{essinf}_{x\in\Omega} a}\right)^{\frac{1}{2}}$ and $B_0 := B_{R_p}(x_0) = \{x \in \Omega : d(x, x_0) < R_p\}$. Since u is a solution of problem $(\mathcal{P}_*)$, fromthe last identity of Corollary 7, u verifies the equation

$$0 = -\Delta u(x) - a(x)\,G_u(u(x)) - \frac{1}{2}\left(G_u^2(u(x))\right)' - b(x)\,p'(u(x)) \quad \text{in } \Omega.$$

Now, by the properties of b, p', we have that $b(x)\,p'(u(x)) \geq 0$ a.e. $x \in \Omega$ and by estimates on $\left(G_u^2(u(x))\right)'$ and u (see Corollary 7) and the fact that u has not flat region, we have that $0 \leq -\Delta u(x) - aG_u(u(x)) + 2\lambda\|b\|_{L^\infty(\Omega)}u_+(x)$ in Ω. Then

$$-\Delta u + f(x,u) \geq 0 \text{ in } B_0 \tag{9}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}^+$ is defined by $f(x,\tau) = -aG_u(\tau) + 2\lambda\|b\|_{L^\infty(\Omega)}\tau_+$.Notice that $f(x,\cdot)$ is non–decreasing in τ since G_u is a non–increasing function. So we can apply the comparison principle for quasi–linear problems (see e.g. [4]). Now, we consider the solution $\underline{u}$ given in Lemma 8 with $B = B_0 = B_{R_p}(x_0)$ and $\hat{a} := F_v \operatorname*{essinf}_{x\in\Omega} a(x)$. So, $\underline{u}$ is verifies

$$(PB)\begin{cases} -\Delta \underline{u}(x) & = F_v \operatorname*{essinf}_{x\in\Omega} a(x) \quad \text{in } B_0, \\ \underline{u}(x) & = \gamma \qquad\qquad\qquad \text{on } \partial B_0 \end{cases}$$

and by property of R_p, $\underline{u}(x) < 0$ for all $x \in B_0 \setminus \{x_0\}$. Notice that, by integration, the exact solution $\underline{u}$ is given by $\underline{u}(x) = \gamma + F_v\left(R_p - |x - x_0|^2\right) \operatorname*{essinf}_{x\in\Omega} a(x)$ for all $x \in B_0$ and $\underline{u}(x_0) = \gamma + F_v R_p \operatorname*{essinf}_{x\in\Omega} a(x) = 0$ from the definition of R_p. On the other hand

$$f(x, \underline{u}) \leq -a(x) F_v - F_v \operatorname*{essinf}_{\Omega} a < 0 \quad \text{a.e. in } B_0 \tag{10}$$

from the assumption of theorem. So, we get that

$$\begin{aligned} -\Delta \underline{u} + f(x, \underline{u}) < 0 \leq -\Delta u + f(x, u) \quad &\text{in } B_0,\\ \underline{u}(x) = \gamma \leq u(x) \quad &\text{on } \partial\Omega \end{aligned}$$

form (9), (10) and the property $u \geq \gamma$ in Ω. By the comparison principle, we conclude that $u \geq \underline{u}$ in $\bar{B}_0 \Rightarrow u(x_0) \geq \underline{u}(x_0) = 0$. Since the solution u has not flat region, we can deduce that $u > 0$ a.e. in $\{x \in \Omega : d(x, \partial\Omega) \geq R_p\}$ and thus $\{x \in \Omega : d(x, \partial\Omega) \geq R_p\} \subset \Omega_p := \{x \in \Omega : u(x) > 0\}$. □

Proof of Theorem 2. Let $\rho = \left(\frac{2S}{K}\right)^{1/2}$ be the positive constant introduced in Theorem 2 with K the bound obtained in Corollary 6. Then $R_v := \hat{R} - \left(\frac{2S}{K}\right)^{1/2}$. We take a point $x_1 \in \Omega$ such that $d(x, \partial\Omega) = R_v$ and a point $\bar{x}_1 \in \partial\Omega$ such that $d(x_1, \partial\Omega) = d(x_1, \bar{x}_1)$. Then $x_1 = \bar{x}_1 + R_v\mathbf{n}$ and $\{x : x = \bar{x}_1 + r\mathbf{n},\ 0 \leq r \leq \hat{R}\} \subset \bar{\Omega}$. On this segment, u satisfies the nonlinear equation

$$-u''(r) = a(r) G_u(u(r)) + \frac{1}{2}\left(G_u^2(u(r))\right)' + b(r) p'(u(r))$$

for $0 < r < \hat{R}$ (here, for a given function $h : \Omega \to \mathbb{R}$, we use the notation $h(r) := h(\bar{x}_1 + r\mathbf{n})$). From Corollary 7, $-u''(r) \leq K$. Moreover $u(0) = u(\bar{x}_1) = \gamma$ and $u(\hat{R}) = u(\bar{x}_1 + \hat{R}\mathbf{n}) \leq \|u_+\|_{L^\infty(\Omega)}$. As in the proof of Theorem 1, we can to find a bound for the right hand side and u verifies the equations $-u''(r) \leq K$ in $(0, \hat{R})$, $u(0) = \gamma$ and $u(\hat{R}) \leq \|u_+\|_{L^\infty(\Omega)} \leq S$. Notice that $u \geq \gamma$ and $\gamma < 0$, thus if $u(\hat{R}) \leq 0$ then $u(\hat{R}) \leq \|u_+\|_{L^\infty(\Omega)}$; and if $u(\hat{R}) > 0$ then $u(\hat{R}) = u_+(\hat{R}) \leq \|u_+\|_{L^\infty(\Omega)}$. Now, we consider the real function $v(r) := S - \frac{1}{2}K(\hat{R} - r)^2$ for $r \in [0, \hat{R}]$. Then, by definition of v, on has that $v(\hat{R}) = S$, $v(R_v) = S - \frac{1}{2}K(\hat{R} - R_v)^2 = 0$ and since v is increasing in $(0, \hat{R})$ then $v(r) \leq 0$ in $(0, R_v)$. One the other hand, v is the unique solution to the linear boundary problem

$$(BP)\left\{\begin{array}{lll} -v''(r) & = K & \text{in } (0, \hat{R}),\\ v(0) & = \gamma, & v(\hat{R}) = S. \end{array}\right.$$

Thus, $u(r) \leq v(r)$ in $(0, \hat{R})$. Moreover, $v(R_v) = 0 > v(r) \geq u(r)$ for any $r \in [0, R_v)$. In particular for all $0 < r < R_v$, one has that $u(\bar{x}_1 + r\mathbf{n}) < 0$ and then the segment $\{x = \bar{x}_1 + r\mathbf{n}, 0 < r < R_v\} \subset \Omega_v = \{x \in \Omega : u(x) < 0\}$. Thus $\{x \in \Omega : d(x, \partial\Omega) \leq \hat{R} - \rho\} \subset \Omega_v = \{x \in \Omega : u(x) < 0\}$. □

Considering the estimates given by the Theorem 1 and Theorem 2 joint the numerical result of [2] we can check theobtained results thanks to the numerical simulation. On the other hand, we can considers these results as a test for the numerical approach.

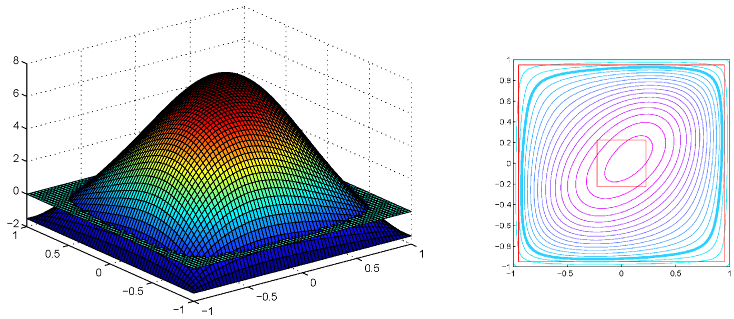

Figure 2: the plasma region, $u > 0$ in blue bold line) joint to the estimate for the distance form the free bounday of the plasma region to the plasma region to the boundary of domain (square in red color).

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